presented previously.[1,2] For some applications, the method of Ref. 2 might be preferred because with it the velocity vector of the moving object can be found without knowing the coordinates of the listening stations. However, at least three such stations are needed. The present Note gives an alternate method of computation in which only two observation posts are required. This can be advantageous in some cases.

Let t be the time at which the shock wave arrives at a particular point. Finite difference approximations for $\partial t/\partial x$, $\partial t/\partial y$ and $\partial t/\partial z$ can be obtained by having two or more microphones along the x, y and z axes through the point, respectively. This gives ∇t, the gradient of t. But the shock wave travels with the speed of sound, C, in the direction perpendicular to the wave front. Hence ∇t is in the direction of the vector $\mathbf{C}$, and its magnitude is $1/C$. That is,

$$C^2 \nabla t = \mathbf{C} \tag{1}$$

$$(\partial t/\partial x)^2 + (\partial t/\partial y)^2 + (\partial t/\partial z)^2 = 1/C^2 \tag{2}$$

If the speed of sound is known, then only two of the partial derivatives of t need be measured. The third one can be calculated, except for sign, by means of Eq. (2). Thus a cluster of microphones can serve as an observation post to determine ∇t, and therefore $\mathbf{C}$.

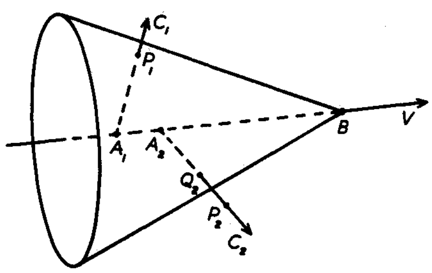

Fig. 1 The object at B, moving with velocity V greater than C, creates a shock cone as shown.

Refer now to Fig. 1. It shows a portion of the shock cone generated by an object moving along path $A_1 A_2 B$ with velocity $\mathbf{V}$. Observers are at P_1 and P_2. The shock wave has just arrived at P_1 and Q_2. The time will be designated as t_1, and the arrival time at P_2 will be called t_2. Hence

$$\mathbf{Q}_2 = \mathbf{P}_2 - \mathbf{C}_2(t_2 - t_1) \tag{3}$$

This gives the point Q_2. Let τ_1 be the time required for the shock wave to travel from A_1 to P_1, and τ_2 the time it spends going from A_2 to Q_2. Then

$$\mathbf{V}(\tau_1 - \tau_2) = \mathbf{C}_1 \tau_1 + \mathbf{Q}_2 - \mathbf{P}_1 - \mathbf{C}_2 \tau_2 \tag{4}$$

But the cosine of the angle between $\mathbf{V}$ and $\mathbf{C}$ is equal to C/V. Therefore $\mathbf{V} \cdot \mathbf{C} = C^2$. So dotting Eq. (4) with $\mathbf{C}_2$ and then $\mathbf{C}_1$ gives

$$(C^2 - \mathbf{C}_1 \cdot \mathbf{C}_2)\tau_1 = \mathbf{C}_2 \cdot (\mathbf{Q}_2 - \mathbf{P}_1) \tag{5}$$

$$(C^2 - \mathbf{C}_1 \cdot \mathbf{C}_2)\tau_2 = \mathbf{C}_1 \cdot (\mathbf{P}_1 - \mathbf{Q}_2) \tag{6}$$

These equations give τ_1 and τ_2 unless $\mathbf{C}_1 = \mathbf{C}_2$. If $(\mathbf{C}_1 + \mathbf{C}_2) \cdot (\mathbf{Q}_2 - \mathbf{P}_1)$ is not zero, then τ_1 and τ_2 will be different, and $\mathbf{V}$ can be obtained from Eq. (4). Also,

$$\mathbf{A}_1 = \mathbf{P}_1 - \mathbf{C}_1 \tau_1 \tag{7}$$

Thus the point A_1 on the path of the moving object, and the time $t_1 - \tau_1$ when it was there, are also known.

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Prediction of Turbulent Boundary-Layer Separation Influenced by Blowing

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Introduction

RESULTS of a study of turbulent boundary-layer separation were presented in Ref. 1. In an effort to obtain additional data on flow separation at high speeds, another test series was carried out in the NOL Hypersonic Tunnel to study the effect of mass addition. The attempt to predict separation of a turbulent boundary layer over a cone-flare configuration with injection by an extension of the method of Ref. 1 resulted in this Note.

Consider the supersonic, axisymmetric turbulent boundary layer over a circular cone which is fed by gas injected normally through a porous section of the wall and subsequently its separation is induced by a conical flare. Separation is assumed to commence over the solid portion of the wall past the porous section. The analysis is concerned primarily with prediction of the turbulent boundary-layer characteristics ahead of separation. When these are known, the separation point is determined by the same procedure as used in Ref. 1. The analysis rests on some concepts introduced in Ref. 2 at low speeds, modified to be applicable to the high-speed flow. It is assumed that: a) the injected gas, having the same composition as the gas in the boundary layer is perfect; b) the blowing rate does not reach values at which the assumptions of the boundary-layer theory are violated.

Turbulent Boundary-Layer Characteristics

For constant flow conditions along the outer edge of the boundary layer the momentum integral equation is

$$c_f/2 = d(x\delta_2)/x\,dx - \rho_w v_w/\rho_e u_e \tag{1}$$

For a turbulent boundary layer with no blowing the skin-friction coefficient, c_f, and the Reynolds number based on momentum thickness, Re_2, are related by

$$c_{fo} Re_{2o}{}^{1/4} = C_1 \tag{2}$$

C_1 is, for given freestream conditions and wall temperature, a constant evaluated by the reference-temperature method. The subscript zero refers to an impermeable wall.

Consider first flow past a cone with permeable wall over its whole length. The boundary layer is assumed to be fully turbulent. From Eqs. (1) and (2) we get

$$2d(Re\,Re_2)/C_1 (Re\,Re_2)^{-1/4} = Re^{5/4}(b+\psi)\,d\,Re \tag{3}$$

where Re denotes the Reynolds number based on distance x, and ψ and the blowing parameter b are defined by

$$\psi \equiv (c_f/c_{fo})_{Re_2}, \quad b \equiv 2\rho_w v_w/\rho_e u_e c_{fo} \tag{4}$$

In definition (4) both c_f and c_{fo} are evaluated at the same freestream conditions and wall temperature but at corresponding stations such that, Re_2, is the same. For constant b and ψ integration of Eq. (3) yields

$$(Re\,Re_2)^{5/4} = \tfrac{5}{18} C_1 Re^{9/4}(b+\psi) \tag{5}$$

Evaluate Eq. (5) at the same value of Re with $b = 0$ and divide this into Eq. (5) to get

$$Re_2 = (b+\psi)^{4/5} Re_{2o} \tag{6}$$

The skin-friction coefficients for no blowing evaluated from Eq. (2)

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Table 1 Comparison of the theoretical and experimental separation data

$Re_1 \times 10^{-6}$, per in.	T_t, °R	P_t, atm	T_w/T_t	Flare angle, deg γ_F	Blowing rate = $\rho_w v_w$, lbm/ft^2-sec	Separation length ΔS, in. Measured	Calculated
3.468	862	132.0	0.600	40	0	...	1.023
4.075	669	100.0	0.600	40	↓	...	1.010
0.962	924	40.0	0.600	35	↓	...	0.615
1.721	948	75.0	0.600	35	↓	...	0.564
1.683	957	74.5	0.675	30	↓	0.36	0.254
3.360	938	143.9	0.648	30	↓	0.30	0.255
3.191	968	143.8	0.666	25	↓	0.08	0.000
1.729	948	75.4	0.666	25	↓	0.10	0.000
0.892	956	39.4	0.657	25	↓	0.10	0.000
1.724	947	74.9	0.576	20	↓	0.00	0.000
3.230	970	144.0	0.567	20	↓	0.00	0.000
3.400	931	143.9	0.567	20	0.250	0.06	0.000
3.380	934	143.8	0.567	20	0.623	0.06	0.000
3.256	955	143.5	0.576	25	0.263	0.08	0.135
3.214	967	144.4	0.558	25	0.617	0.10	0.273
3.313	942	142.8	0.594	30	0.228	0.38	0.442
3.191	968	143.8	0.576	30	0.608	1.00	0.811

at Re and Re_2 are $(c_{fo})_{Re} = C_1 Re_{2o}^{-1/4}$ and $(c_{fo})_{Re_2} = C_1 Re_2^{-1/4}$. And because $(c_f)_{Re} = (c_f)_{Re_2} = c_f$, Eq. (6) yields

$$(c_f/c_{fo})_{Re} = \psi(b+\psi)^{-1/5} \tag{7}$$

The subscript Re in Eq. (7) indicates that both c_f and c_{fo} are evaluated at the same local Reynolds number (at the same stations, x, since the freestream conditions are the same). The form of this equation is identical with that derived in Ref. 2 for the incompressible flow over a flat plate. In Ref. 2 we also find $(\bar{c}_f/\bar{c}_{fo})_{Re_2} \equiv \bar{\psi} = (1-\bar{b}/4)^2$. The barred quantities refer to the incompressible case. In analogy to the reference-temperature method, where it is assumed that the functional forms expressing the dependence of c_f on Re_2 are similar for both compressible and incompressible cases, we write

$$\psi = (1-b/4)^2 \tag{8}$$

The local skin-friction coefficient is obtained from Eqs. (7) and (8) where c_{fo} is calculated by the application of the reference-temperature method [Eq. (2)]. That the relation (8) is plausible can be inferred by comparing it to Eq. (4.30) of Ref. 2. It should be observed that the definitions of b and ψ differ in the present Note from those in Ref. 2. Denoting by subscript k the quantities in Ref. 2 we have $\psi_k \equiv (c_f/\bar{c}_{fo})_{Re_2} = \psi(c_{fo}/\bar{c}_{fo})_{Re_2}$ and $b_k \equiv 2\rho_w v_w/\rho_e u_e \bar{c}_{fo} = b(c_{fo}/\bar{c}_{fo})_{Re_2}$. These relations combine with Eq. (8) to give

$$\psi_k = [(c_f/\bar{c}_{fo})_{Re_2}][1-(b_k/4)/(c_{fo}/\bar{c}_{fo})_{Re_2}]^2 \tag{9}$$

Equation (9) resembles the semiempirical relation (4.30) of Ref. 2 which was found to be valid for subsonic boundary layers. By a more rigorous approach we shall now examine the supposition represented by Eq. (8). Lewis[3] has expanded Coles' theory in which a correspondence between the high-speed flow and incompressible boundary-layer flow is established. To examine how Eq. (8) stands up in light of the results of Refs. 3 and 4 is of interest. For $\rho\mu$ = constant and p = constant it was found [cf. Ref. (3)] that at corresponding stations $c_f Re_2 = \bar{c}_f \bar{R}e_2$ and $(\rho_w v_w/\rho_e u_e)Re_2 = (\bar{\rho}_w \bar{v}_w/\bar{\rho}_e \bar{u}_e)\bar{R}e_2$. Combine these relations and multiply the resulting equation by $(\bar{c}_{fo})_{\bar{R}e_2}/(c_{fo})_{Re_2}$ to get

$$\psi/\bar{\psi} \equiv (c_f/c_{fo})_{Re_2}/(\bar{c}_f/\bar{c}_{fo})_{Re_2} = b/\bar{b} \tag{10}$$

Since the relation $c_f Re_2 = \bar{c}_f \bar{R}e_2$ applies to both the blowing and no blowing case,[3] we may also write

$$\frac{(c_{fo})_{Re_2}}{(\bar{c}_{fo})_{\bar{R}e_2}} = \frac{(c_{fo})_{Re_{2o}}}{(\bar{c}_{fo})_{\bar{R}e_2}} \frac{(c_{fo})_{Re_2}}{c_{fo/Re_{2o}}} = \frac{\bar{R}e_2}{Re_{2o}} \frac{(c_{fo})_{Re_2}}{(c_{fo})_{Re_{2o}}} \tag{11}$$

where Re_{2o} and Re_2 are the Reynolds numbers at corresponding stations $\bar{x}$ and x, respectively, for the case of no blowing. Combining $c_f Re_2 = \bar{c}_f \bar{R}e_2$ with Eq. (11) we get

$$\psi/\bar{\psi} = [(c_f/c_{fo})_{Re_2}/(\bar{c}_f/\bar{c}_{fo})_{\bar{R}e_2}] = (c_{fo})_{Re_{2o}} Re_{2o}/(c_{fo})_{Re_2} Re_2 \tag{12}$$

It is known that for laminar flow at constant pressure

$$(c_{fo})_{Re_2} Re_2 = (c_{fo})_{Re_{2o}} Re_{2o} = \text{const} \tag{13}$$

Hence, $\psi = \bar{\psi}$ and by Eq. (10) $b = \bar{b}$ at corresponding stations. So that for laminar flow Eq. (10) reduces to Eq. (8). For turbulent flow Eq. (13) does not apply and, therefore, Eqs. (8) and (10) will not be congruent. From the rigorous point of view this is subject to the same criticism as the widely used reference-temperature method (see p. 1418, Ref. 4).

For a partly porous wall we propose to treat the transition from solid to porous section (and vice versa) as the transition from laminar to turbulent flow is treated; the growth of the momentum thickness will be assumed to be identical to that of a fully turbulent boundary layer developed over a fully porous wall starting at some fictitious origin. When Eq. (3) is integrated from the point where blowing starts ($x = x_i$) to the station x behind the porous section ahead of separation holding b constant, it yields a relation for Re_2,

$$(Re\,Re_2)^{5/4} - (Re_i\,Re_{2i})^{5/4} = \tfrac{5}{18}C_1[(Re^{9/4} - Re_j^{9/4}) + (Re_j^{9/4} - Re_i^{9/4})(b+\psi)] \tag{14}$$

for $x > x_j$, where $x = x_j$ is the location where blowing ends. The skin-friction coefficient can be calculated from Eqs. (2) and (14).

Separation Prediction and Discussion of Results

Since the blowing terminates ahead of separation, the analysis of the separated region is very similar to that presented in Ref. 1 for an impermeable wall. The resulting relations[5] differ slightly from those obtained in Ref. 1. The difference is due only to a change in geometry.

The agreement between theoretical predictions and test data for both blowing and no blowing is favorable (Table 1). Experimental data for the first four cases were not available. The data refer to a fore cone of 20-in. length and 4.5° semiangle, and Mach number 6. The separation length is measured along the cone surface from the separation point to the cone-flare junction. In the experiments the blowing rate was uniform. The assumption of constancy of b in derivation of Eqs. (7) and (14) precludes the case of uniform blowing. However, for $\bar{\rho}_w \bar{v}_w$ = const. Reference 2 shows small deviation from Eq. (7). Because separation takes place past the point where injection terminates, the separation lengths even at higher blowing rates are not excessively large in comparison to no blowing. From the first two entries in Table 1 a drop in separation length with increasing Reynolds number at high Reynolds number is observed. The same finding was reported previously in Ref. 1.

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Computational Aspects of Asymptotic Matching in the Restricted Three-Body Problem

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THE results of asymptotic matching in the restricted three-body problem yield an asymptotic approximation to the perturbed hyperbola about the moon in an Earth-to-moon trajectory.[1–4] The approximation is given in terms of the initial conditions, small variations in the initial conditions and definite integrals of bounded continuous functions. The definite integrals, however, involve the bounded difference of two quantities which themselves become unbounded at the upper limit of integration. The purpose of this Note is to describe the technique used to compute these definite integrals.

Let $\boldsymbol{r}(t, \mu)$ be the solution of the restricted three-body problem

$$\ddot{\boldsymbol{r}} = \boldsymbol{F}(\boldsymbol{r}) + \mu \boldsymbol{f}(\boldsymbol{r}, t), \quad \mu \ll 1$$

$$\boldsymbol{r}(t_0) = \boldsymbol{r}_0(t_0) + \delta\boldsymbol{r}_0, \quad \dot{\boldsymbol{r}}(t_0) = \dot{\boldsymbol{r}}(t_0) + \delta\dot{\boldsymbol{r}}_0 \tag{1}$$

where

$$\boldsymbol{F}(\boldsymbol{r}) = -\boldsymbol{r}/|\boldsymbol{r}|^3$$

$$\boldsymbol{f}(\boldsymbol{r}, t) = -\left[\frac{\boldsymbol{r} - \boldsymbol{r}_m(t)}{|\boldsymbol{r} - \boldsymbol{r}_m(t)|^3} + \frac{\boldsymbol{r}_m(t)}{|\boldsymbol{r}_m(t)|^3} - \frac{\boldsymbol{r}}{|\boldsymbol{r}|^3}\right]$$

and where $\boldsymbol{r}_m(t)$, the moon's position at time t, satisfies

$$\ddot{\boldsymbol{r}}_m = \boldsymbol{F}(\boldsymbol{r}_m) \tag{2}$$

with the initial conditions $\boldsymbol{r}_m(t_0)$ and $\dot{\boldsymbol{r}}_m(t_0)$ chosen so that $\boldsymbol{r}_m(t) \neq 0$ for $t \geq t_0$. Let $\boldsymbol{r}_0(t)$ be a solution of the two-body problem

$$\ddot{\boldsymbol{r}}_0 = \boldsymbol{F}(\boldsymbol{r}_0) \tag{3}$$

with the initial conditions $\boldsymbol{r}_0(t_0)$ and $\dot{\boldsymbol{r}}_0(t_0)$ chosen so that $r_0(t) \neq 0$ and so that at $t_1 > t_0$

$$\boldsymbol{r}_0(t_1) = \boldsymbol{r}_m(t_1)$$

and

$$\boldsymbol{V}_1 \equiv \dot{\boldsymbol{r}}_0(t_1) - \dot{\boldsymbol{r}}_m(t_1) \neq 0$$

i.e., the initial conditions $\boldsymbol{r}_0(t_0)$ and $\dot{\boldsymbol{r}}_0(t_0)$ are chosen so that the conics $\boldsymbol{r}_0$ and $\boldsymbol{r}_m(t)$ intersect at a time $t_1 > t_0$ with nonzero relative velocity. It is assumed that t_1 is the first such time; i.e., it is assumed that $\boldsymbol{r}_0(t) \neq \boldsymbol{r}_m(t)$ for $t_0 \leq t < t_1$. (Note that in moon-to-moon orbits it is assumed that $\boldsymbol{r}_0(t_0) = \boldsymbol{r}_m(t_0)$ and that $\boldsymbol{r}_0(t_1) = \boldsymbol{r}_m(t_1)$; however, moon-to-moon orbits which are treated in Ref. 1 will not be considered here.)

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Under the above assumptions, the method of asymptotic matching[1–4] can be applied to the restricted three-body problem in order to obtain an asymptotic approximation to the solution $\boldsymbol{r}(t, \mu)$ which, in the case of first-order matching, has been shown to be uniformly valid to $0(\mu^2)$ during one moon passage; i.e., over a time interval $[t_0, t_1 + 0(\mu)]$; cf. Refs. 4 and 5.

In the method of asymptotic matching in the restricted three-body problem one first expands the solution $\boldsymbol{r}(t, \mu)$ of Eq. (1) relative to the Earth in an asymptotic series (called the outer expansion)

$$\boldsymbol{r}(t, \mu) \simeq \boldsymbol{r}_0(t) + \mu\boldsymbol{\rho}_1(t) + \mu^2\boldsymbol{\rho}_2(t) + \cdots$$

where $\boldsymbol{r}_0(t)$ is the solution of the two-body problem (3) and it is shown in Ref. (1) that the first-order perturbation

$$\boldsymbol{\rho}_1(t) = \boldsymbol{\rho}_{1b}(t) + \boldsymbol{i}V_1^{-2}\ln(\tau_0/\tau)$$

where the bounded part of the first-order perturbation

$$\boldsymbol{\rho}_{1b}(t) = \Phi_{rr}(t, t_0)\delta\boldsymbol{r}_0/\mu + \Phi_{rv}(t, t_0)\delta\dot{\boldsymbol{r}}_0/\mu - \boldsymbol{i}[1 - (\tau/\tau_0)]/V_1^2$$
$$+ \int_{t_0}^{t} \{\Phi_{rv}(t, t')\boldsymbol{f}[\boldsymbol{r}_0(t'), t'] - \boldsymbol{i}(t - t')/(V_1^2\tau'^2)\}dt'$$

with $\boldsymbol{i} = \boldsymbol{V}_1/|\boldsymbol{V}_1|$, $\tau = t_1 - t$, $\tau' = t_1 - t'$ and $\tau_0 = t_1 - t_0$. The transition matrix

$$\Phi = \begin{bmatrix} \Phi_{rr} & \Phi_{rv} \\ \Phi_{vr} & \Phi_{vv} \end{bmatrix}$$

satisfies

$$\dot{\Phi}(t, \cdot) = \begin{bmatrix} 0 & I \\ d\boldsymbol{F}/d\boldsymbol{r}[\boldsymbol{r}_0(t)] & 0 \end{bmatrix}\Phi(t, \cdot) \tag{4}$$

and

$$\Phi(t, t) = I$$

where I is the identity matrix.

Next, one expands the solution

$$\boldsymbol{\rho}^*(t, \mu) = \boldsymbol{r}(t, \mu) - \boldsymbol{r}_m(t)$$

of the restricted three-body problem

$$\ddot{\boldsymbol{\rho}}^* = \mu \boldsymbol{F}(\boldsymbol{\rho}^*) - (1 - \mu)\left[\frac{\boldsymbol{\rho}^* + \boldsymbol{r}_m(t)}{|\boldsymbol{\rho}^* + \boldsymbol{r}_m(t)|^3} - \frac{\boldsymbol{r}_m(t)}{|\boldsymbol{r}_m(t)|^3}\right]$$

relative to the moon in an asymptotic series (called the inner expansion)

$$\boldsymbol{\rho}^*(t, \mu) = \boldsymbol{\rho}_0^*(t, \mu) + \boldsymbol{\rho}_1^*(t, \mu) \cdots$$

where $\boldsymbol{\rho}_0^*(t, \mu)$ satisfies the two-body equations

$$\ddot{\boldsymbol{\rho}}_0^* = \mu \boldsymbol{F}(\boldsymbol{\rho}_0^*)$$

and describes a moon centered hyperbola with parameters Δ, the distance to the asymptote of the hyperbola V_∞, the velocity at infinity on the hyperbola and t_p the time of perilune passage on the hyperbola.

The parameters of the moon centered hyperbola are determined in terms of the initial conditions and small variations in the initial conditions $\boldsymbol{r}_0(t_0)$, $\dot{\boldsymbol{r}}_0(t_0)$, $\delta\boldsymbol{r}_0$, and $\delta\dot{\boldsymbol{r}}_0$ by asymptotic matching which is carried out to first-order in Refs. 1 and 4 [including the computation of the first-order perturbation to the moon centered hyperbola $\boldsymbol{\rho}_1^*(t, \mu)$ which contributes $O(\mu^{3/2})$ terms to the first-order matching]. The asymptotic matching compares the asymptotic behavior of the inner and outer expansions in the matching region in order to determine the parameters of the moon centered hyperbola. The matching region is defined as the region where the errors in the inner and outer expansions are of the same order. This is shown in Refs. 4 and 5 to take place when $\tau = O(\mu^{1/2})$; i.e., when $|\boldsymbol{\rho}_0^*(t)| = O(\mu^{1/2})$; when the particle is at a distance of $O(\mu^{1/2})$ from the moon, the Earth-moon distance at t_1, $|\boldsymbol{r}_m(t_1)|$, being normalized to order one. The results of the asymptotic matching for the planar problem imply that